Mathematics and Computing/Technology
MT365 Graphs, networks and design

MT365 ATN2

The Open University

MT365
Audio Notes 2

CDA 5675 from TRACK 9 pp 4 – 19
CDA 5676 pp 20 – end

3.4

SUP 54568 7

This booklet contains the printed material for use with Audio-tape 3 for the audio-tape sections of *Networks 3* and Audio-tape 4 for the audio-tape sections of *Graphs 4*.

You will need to play the tape at the same time as you study the frames on the following pages.

Place the cassette player within easy reach. There are points on the tape where we have indicated that we want you to stop the tape and do some work for yourself, but you will probably find it necessary to stop the tape more often than this. Indeed, you should get into the habit of frequently stopping the tape and giving yourself time to think.

Make sure that you have paper and pencil handy before starting each tape sequence.

BLOCK 3

Notes for Networks 3

There are three sequences on the tape for this unit. The heading numbers below refer to the corresponding sections in *Networks 3*.

In each tape sequence we demonstrate the use of an algorithm to solve an example. In the first two tape sequences we then ask you to use the algorithm to solve a problem. There is a problem on the third algorithm in the text of the unit. Additional problems requiring the use of these three algorithms are given in the *Computer Activities Booklet*.

Each algorithm involves finding matchings in bipartite graphs, and is based on the idea of an *alternating path*, introduced in Section 2. As a reminder, this definition is given below.

Definition

Let G be a bipartite graph in which the set of vertices is divided into two disjoint subsets X and Y. An **alternating path** with respect to a matching M in G is a path which satisfies the conditions:

(a) the path joins a vertex x in X to a vertex y in Y;

(b) the initial and final vertices x and y are not incident with an edge in M;

(c) alternate edges of the path are in M, and the other edges are not in M.

2.2 Maximum matching algorithm

3.1 Hungarian algorithm for the assignment problem

4.1 Hungarian algorithm for the transportation problem

A formal statement of each algorithm is given in the unit. **You should have the unit open at the appropriate page whilst listening to each tape sequence.**

Page 14 has been left blank to enable other frames to face each other.

BLOCK 4

Notes for Graphs 4

There are two sequences on the tape, both associated with Section 5. The numbers 5.3 and 5.4 below refer to the corresponding sections in *Graphs 4*.

In each sequence we describe an algorithm for solving a particular problem using a branch and bound method. The problems are:

5.3 the knapsack problem;

5.4 the travelling salesman problem.

Each algorithm involves a search for an optimum solution based on the following ideas:

- a branching tree for structuring the search;
- successive improvement of a lower bound for a number to be determined.

In the case of the knapsack problem, the number to be determined is the *total value of items packed*.

In the case of the travelling salesman problem, the number to be determined is the *total length of the route*.

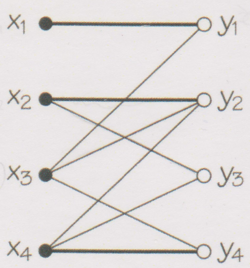
TRACK 9

The maximum matching algorithm

TRACK 10

1 WORKED PROBLEM

Find a maximum matching in the following bipartite graph.



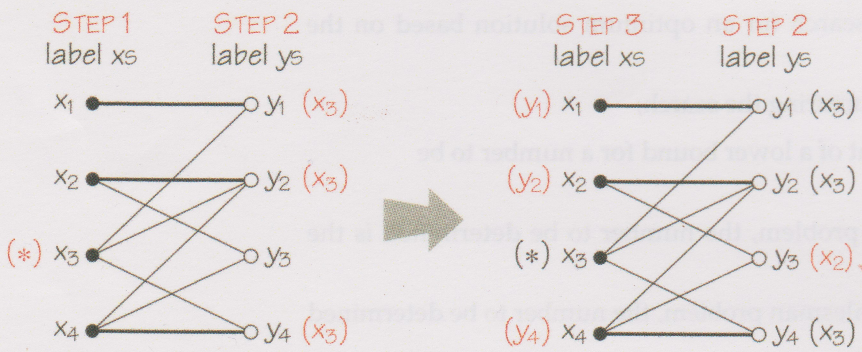
USE

- labelling procedure
- matching improvement procedure

TRACK 11

2 SOLUTION TO WORKED PROBLEM

PART A: LABEL VERTICES



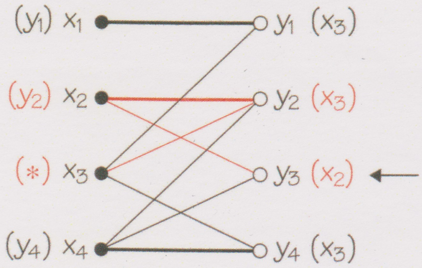
initial matching M

breakthrough at y_3

alternatively, could label y_3 with x_4

PART B: IMPROVE MATCHING

STEP 4: FIND ALTERNATING PATH

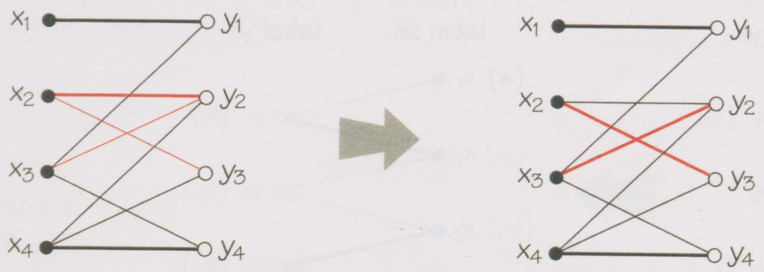


an alternating path is $y_3 x_2 y_2 x_3$

alternatively: $y_3 x_4 y_4 x_3$

2 SOLUTION CONTINUED

STEP 5: CONSTRUCT REVISED MATCHING



original matching
x_1y_1, x_2y_2, x_4y_4

revised matching
x_1y_1, x_4y_4, x_2y_3, x_3y_2

alternatively:
x_1y_1, x_2y_2, x_3y_4, x_4y_3

Since the revised matching has 4 edges, it is a maximum matching.

TRACK 12

3 SUMMARY OF THE ALGORITHM

START with any matching.

Part A: labelling procedure

Label the vertices to identify an alternating path.
If *breakthrough* is achieved, go to Part B.
If *breakthrough* is not achieved, STOP:
the current matching is a maximum matching.

breakthrough occurs when a vertex in Y not incident with any edge in the current matching is labelled

Part B: matching improvement procedure

Find an alternating path by tracing back through the labels.

Form a new matching from:

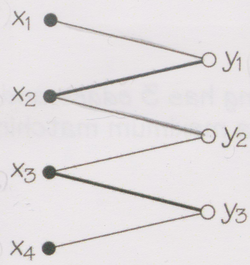
- the edges in the current matching NOT IN the alternating path,
- the edges in the alternating path NOT IN the current matching.

Return to Part A.

TRACK 13

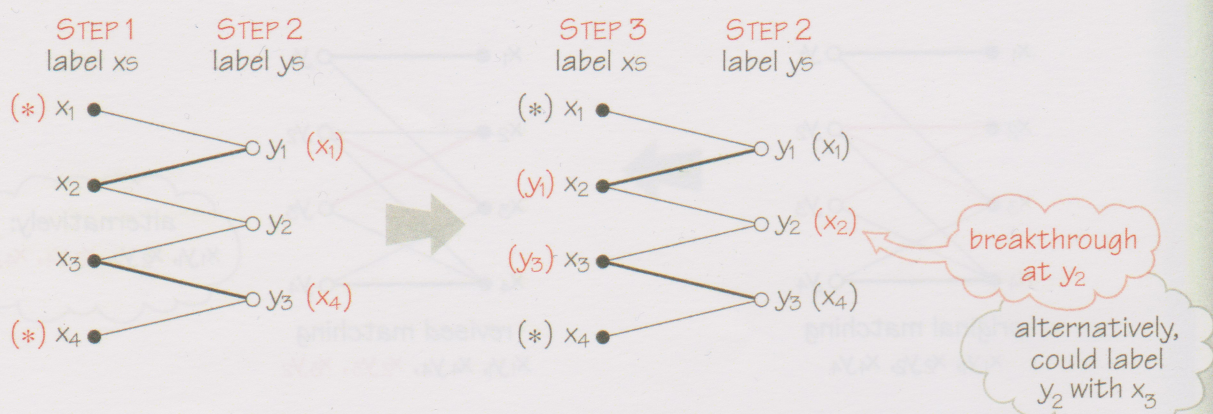
4 PROBLEM

Find an improved matching in the following bipartite graph.



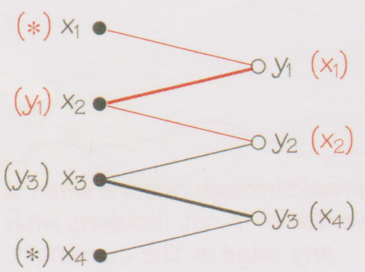
5 SOLUTION

PART A: LABEL VERTICES



PART B: IMPROVE MATCHING

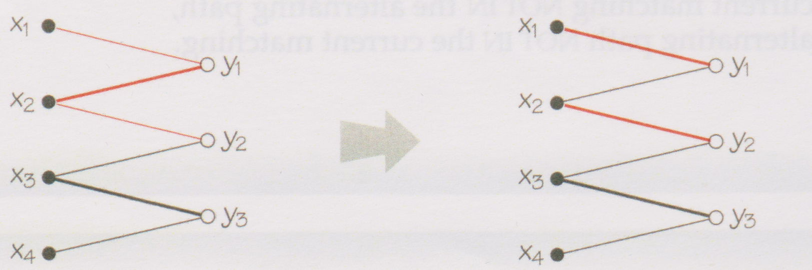
STEP 4: FIND ALTERNATING PATH



an alternating path is
$y_2x_2y_1x_1$

alternatively:
$y_2x_3y_3x_4$

STEP 5: CONSTRUCT REVISED MATCHING



initial matching
x_2y_1, x_3y_3

revised matching
x_3y_3, x_1y_1, x_2y_2

alternatively:
x_2y_1, x_3y_2, x_4y_3

Since the revised matching has 3 edges and there are only 3 vertices in Y, it is a maximum matching.

TRACK 15

The Hungarian algorithm for the assignment problem

USE
- labelling procedure
- matching improvement procedure
- modification of partial graph procedure

1 WORKED PROBLEM

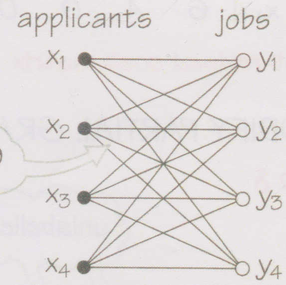
Find the optimum assignment in the following case.

jobs

applicants		y_1	y_2	y_3	y_4
	x_1	6	12	15	15
	x_2	4	8	9	11
	x_3	10	5	7	8
	x_4	12	10	6	9

cost matrix

cost 9



bipartite graph $K_{4,4}$

TRACK 16

2 SOLUTION TO WORKED PROBLEM

STEP 0: CONSTRUCT INITIAL PARTIAL GRAPH

weights ↓

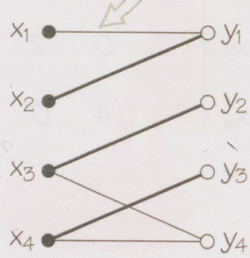
weights		y_1	y_2	y_3	y_4
6	x_1	0	6	9	9
4	x_2	0	4	5	7
5	x_3	5	0	2	3
6	x_4	6	4	0	3

weights →		0	0	0	3
↓		y_1	y_2	y_3	y_4
6	x_1	0	6	9	6
4	x_2	0	4	5	4
5	x_3	5	0	2	0
6	x_4	6	4	0	0

6 zeros

first revised cost matrix

6 edges



maximum matching obtained by inspection

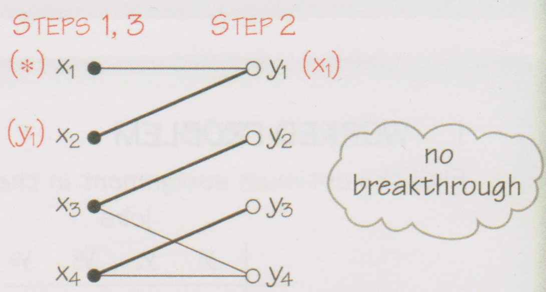
first partial graph

2 SOLUTION CONTINUED

PART A: LABEL VERTICES

		0	0	0	3
		y_1	y_2	y_3	y_4
6	x_1	0	6	9	6
4	x_2	0	4	5	4
5	x_3	5	0	2	0
6	x_4	6	4	0	0

first revised cost matrix



labelled partial graph

PART C: MODIFY PARTIAL GRAPH

STEP 6: FIND δ

unlabelled ys

labelled xs

		0	0	0	3
		y_1	y_2	y_3	y_4
6	x_1	0	6	9	6
4	x_2	0	4	5	4
5	x_3	5	0	2	0
6	x_4	6	4	0	0

first revised cost matrix

unlabelled ys

labelled xs

	y_2	y_3	y_4
x_1	6	9	6
x_2	4	5	4

$\delta = 4$

STEP 7: REVISE COST MATRIX AND PARTIAL GRAPH

2 decrease by δ

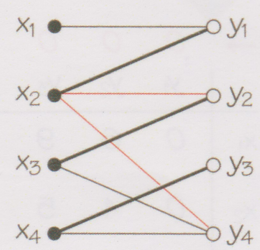
3 decrease by δ

1 increase by δ

		−4	0	0	3
		y_1	y_2	y_3	y_4
10	x_1	0	2	5	2
8	x_2	0	0	1	0
5	x_3	9	0	2	0
6	x_4	10	4	0	0

4 increase by δ

second revised cost matrix

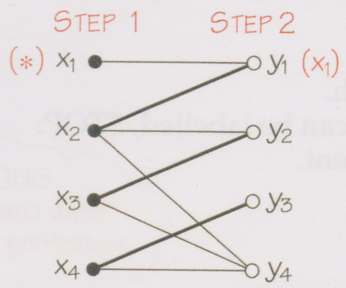


second partial graph

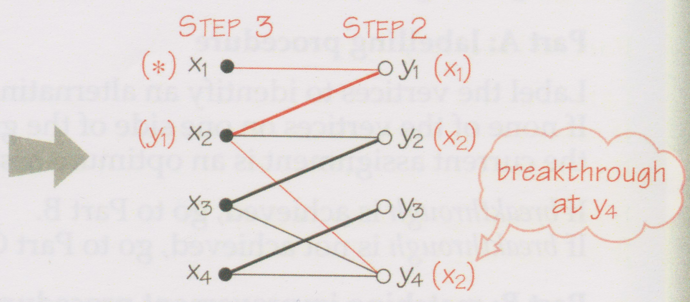
TRACK 18

2 SOLUTION CONTINUED

PART A: LABEL VERTICES



second partial graph

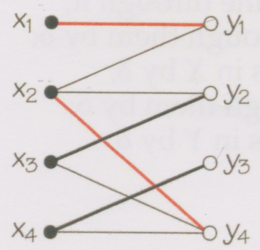


labelled partial graph

PART B: IMPROVE MATCHING

STEP 4: an alternating path is $y_4x_2y_1x_1$.

STEP 5: revised matching is:



new matching
x_3y_2, x_4y_3, x_1y_1, x_2y_4

PART A: LABEL VERTICES impossible STOP

	y_1	y_2	y_3	y_4
x_1	6	12	15	15
x_2	4	8	9	11
x_3	10	5	7	8
x_4	12	10	6	9

original cost matrix

		−4	0	0	3
		y_1	y_2	y_3	y_4
10	x_1	0	2	5	2
8	x_2	0	0	1	0
5	x_3	9	0	2	0
6	x_4	10	4	0	0

final revised cost matrix

optimum assignment: x_1y_1, x_2y_4, x_3y_2, x_4y_3

total cost: $6 + 11 + 5 + 6 = 28 = 10 + 8 + 5 + 6 - 4 + 0 + 0 + 3$

TRACK 19

3 SUMMARY OF THE ALGORITHM

START with no matching.

Assign weights to the vertices and construct the first partial graph.

Part A: labelling procedure

Label the vertices to identify an alternating path.
If none of the vertices on one side of the graph can be labelled, STOP:
the current assignment is an optimum assignment.

SHORT CUT
first time only – find
matching by inspection

If *breakthrough* is achieved, go to Part B.
If *breakthrough* is not achieved, go to Part C.

Part B: matching improvement procedure

Find an alternating path by tracing back through the labels.
Form a new matching.
Return to Part A.

Part C: modification of the partial graph procedure

Construct a revised cost matrix as follows.

On the existing cost matrix:
draw a *horizontal* line through each labelled vertex in X;
draw a *vertical* line through each labelled vertex in Y;
- find the smallest entry δ with ONLY a *horizontal* line through it;
- *decrease* all entries with ONLY a *horizontal* line through them by δ; *increase* the weight on the corresponding vertices in X by δ;
- *increase* all entries with ONLY a *vertical* line through them by δ; *decrease* the weight on the corresponding vertices in Y by δ.

Construct a revised partial graph.
(Remove any edge that now has a non-zero cost.)

Return to Part A.

TRACK 20

4 PROBLEM

Find the optimum assignment in the following case.

	y_1	y_2	y_3	y_4
x_1	8	4	6	7
x_2	10	12	8	14
x_3	9	6	11	15
x_4	12	8	14	8

cost matrix

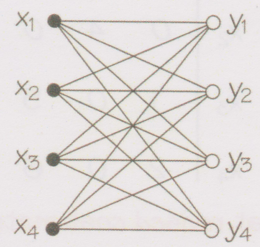
TRACK 21

5 SOLUTION

STEP 0: CONSTRUCT INITIAL PARTIAL GRAPH

	y_1	y_2	y_3	y_4
x_1	8	4	6	7
x_2	10	12	8	14
x_3	9	6	11	15
x_4	12	8	14	8

cost matrix



bipartite graph $K_{4,4}$

weights ↓		y_1	y_2	y_3	y_4
4	x_1	4	0	2	3
8	x_2	2	4	0	6
6	x_3	3	0	5	9
8	x_4	4	0	6	0

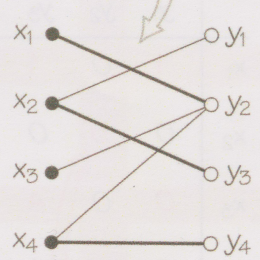
halfway cost matrix

weights →		2	0	0	0
↓		y_1	y_2	y_3	y_4
4	x_1	2	0	2	3
8	x_2	0	4	0	6
6	x_3	1	0	5	9
8	x_4	2	0	6	0

6 zeros

first revised cost matrix

6 edges



matching obtained by inspection

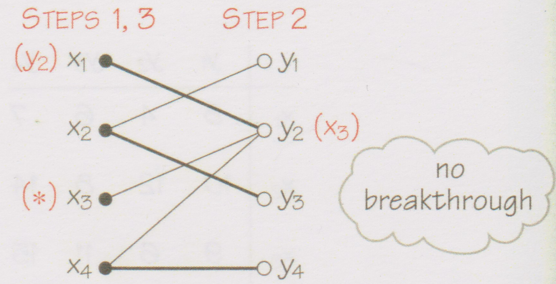
first partial graph

5 SOLUTION CONTINUED

PART A: LABEL VERTICES

		2	0	0	0
		y_1	y_2	y_3	y_4
4	x_1	2	0	2	3
8	x_2	0	4	0	6
6	x_3	1	0	5	9
8	x_4	2	0	6	0

first revised cost matrix



labelled partial graph

PART C: MODIFY PARTIAL GRAPH

STEP 6: FIND δ

unlabelled ys

labelled xs

		2	0	0	0
		y_1	y_2	y_3	y_4
4	x_1	2	0	2	3
8	x_2	0	4	0	6
6	x_3	1	0	5	9
8	x_4	2	0	6	0

first revised cost matrix

unlabelled ys

labelled xs

	y_1	y_3	y_4
x_1	2	2	3
x_3 3	1	5	9

$\delta = 1$

STEP 7: REVISE COST MATRIX AND PARTIAL GRAPH

2 decrease by δ

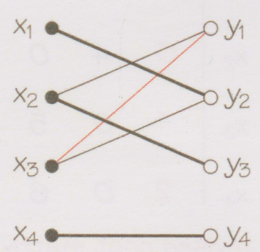
1 increase by δ

3 decrease by δ

		2	−1	0	0
		y_1	y_2	y_3	y_4
5	x_1	1	0	1	2
8	x_2	0	5	0	6
7	x_3	0	0	4	8
8	x_4	2	1	6	0

4 increase by δ

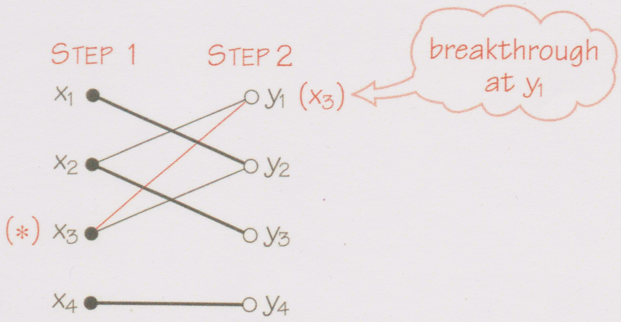
second revised cost matrix



second partial graph

5 SOLUTION CONTINUED

PART A: LABEL VERTICES

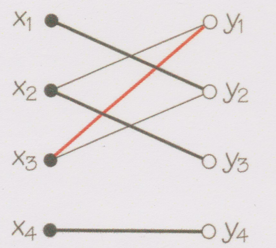


labelled partial graph

PART B: IMPROVE MATCHING

STEP 4: an alternating path is y_1x_3

STEP 5: revised matching is:



new matching
x_1y_2, x_2y_3, x_4y_4, x_3y_1

PART A: LABEL VERTICES

impossible

STOP

	y_1	y_2	y_3	y_4
x_1	8	4	6	7
x_2	10	12	8	14
x_3	9	6	11	15
x_4	12	8	14	8

original cost matrix

		2	−1	0	0
		y_1	y_2	y_3	y_4
5	x_1	1	0	1	2
8	x_2	0	5	0	6
7	x_3	0	0	4	8
8	x_4	2	1	6	0

final revised cost matrix

optimum assignment: x_1y_2, x_2y_3, x_3y_1, x_4y_4

total cost: $4 + 8 + 9 + 8 = 29 = 5 + 8 + 7 + 8 + 2 - 1 + 0 + 0$

TRACK 23

The Hungarian algorithm for the transportation problem

USE
- labelling procedure
- flow-augmenting procedure
- modification of partial graph procedure

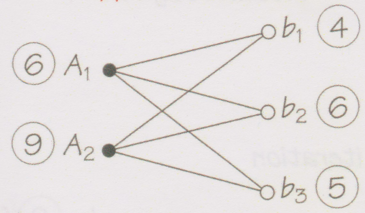
1 WORKED PROBLEM

Find a minimum-cost solution to the transportation problem for the following situation.

demand

			4	6	5
			b_1	b_2	b_3
supply	6	A_1	3	5	7
	9	A_2	6	4	3

cost matrix



bipartite graph

TRACK 24

2 SOLUTION TO WORKED PROBLEM

STEP 0: CONSTRUCT INITIAL PARTIAL GRAPH

weights ↓

			4	6	5
			b_1	b_2	b_3
3	6	A_1	0	2	4
3	9	A_2	3	1	0

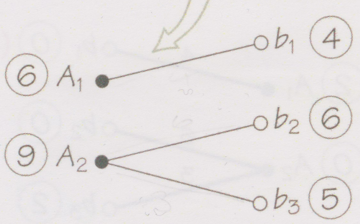
weights → ↓

			0	1	0
			4	6	5
			b_1	b_2	b_3
3	6	A_1	0	1	4
3	9	A_2	3	0	0

3 zeros

first revised cost matrix

3 edges



first partial graph

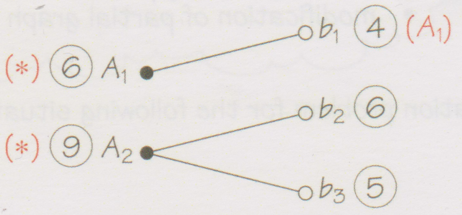
TRACK 25

2 SOLUTION CONTINUED

PART A: LABEL VERTICES

STEPS 1–3

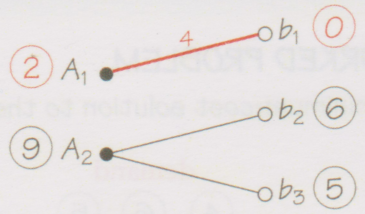
First iteration



breakthrough at b_1

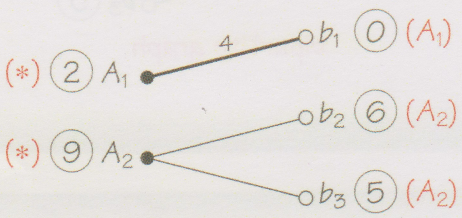
PART B: AUGMENT FLOW

STEPS 4, 5

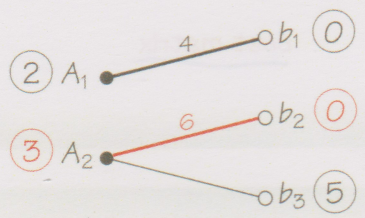


- flow-augmenting path is A_1b_1
- min (6, 4) = 4
- send flow of 4 down A_1b_1

Second iteration

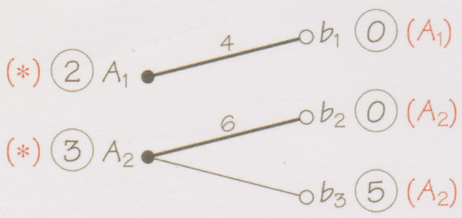


breakthrough at b_2

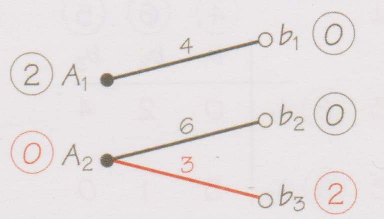


- flow-augmenting path is A_2b_2
- min (9, 6) = 6
- send flow of 6 down A_2b_2

Third iteration

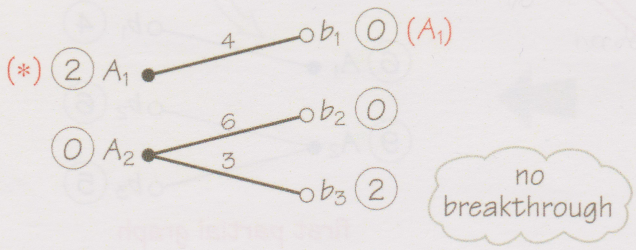


breakthrough at b_3



- flow-augmenting path is A_2b_3
- min (3, 5) = 3
- send flow of 3 down A_2b_3

Fourth iteration



2 SOLUTION CONTINUED

PART C: MODIFY PARTIAL GRAPH

STEP 6: FIND δ

unlabelled demands

labelled supply

	b_1	b_2	b_3
A_1	0	1	4
A_2	3	0	0

first revised cost matrix

unlabelled demands

labelled supply

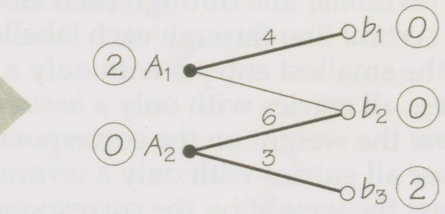
	b_2	b_3
A_1	1	4

$\delta = 1$

STEP 7: REVISE COST MATRIX AND PARTIAL GRAPH

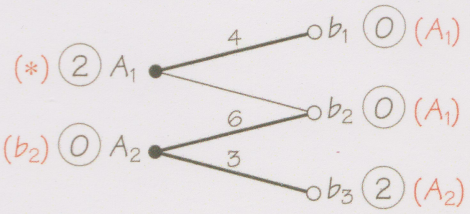
			−1	1	0
			0	0	2
			b_1	b_2	b_3
4	2	A_1	0	0	3
3	0	A_2	4	0	0

second revised cost matrix



second partial graph

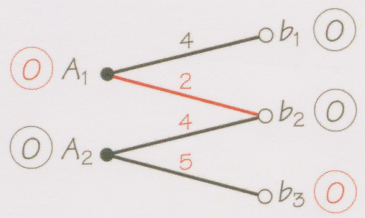
PART A: LABEL VERTICES



breakthrough at b_3

maximum backward flow along A_2b_2 is 6

PART B: AUGMENT FLOW



- flow-augmenting path is $b_3A_2b_2A_1$
- min (6, 2, 2) = 2
- increase flow by 2 on A_1b_2, A_2b_3
- decrease flow by 2 on A_2b_2

final flow is	4 along A_1b_1	with cost	(4×3)
	2 along A_1b_2		$+ (2 \times 5)$
	4 along A_2b_2		$+ (4 \times 4)$
	5 along A_2b_3		$+ (5 \times 3) = 53$

TRACK 27

3 SUMMARY OF THE ALGORITHM

START with no flow.

Construct the initial partial graph.

Part A: labelling procedure

Label the vertices to identify a flow-augmenting path.
If no labelling is possible, STOP:
the current solution is a minimum-cost solution.

If *breakthrough* is achieved, go to Part B.
If *breakthrough* is not achieved, go to Part C.

breakthrough occurs when a demand vertex is labelled whose demand is not satisfied

Part B: flow-augmenting procedure

Find a flow-augmenting path by tracing back through the labels.
Augment the flow.
Return to Part A.

Part C: modification of the partial graph procedure

Construct a new revised cost matrix as follows.

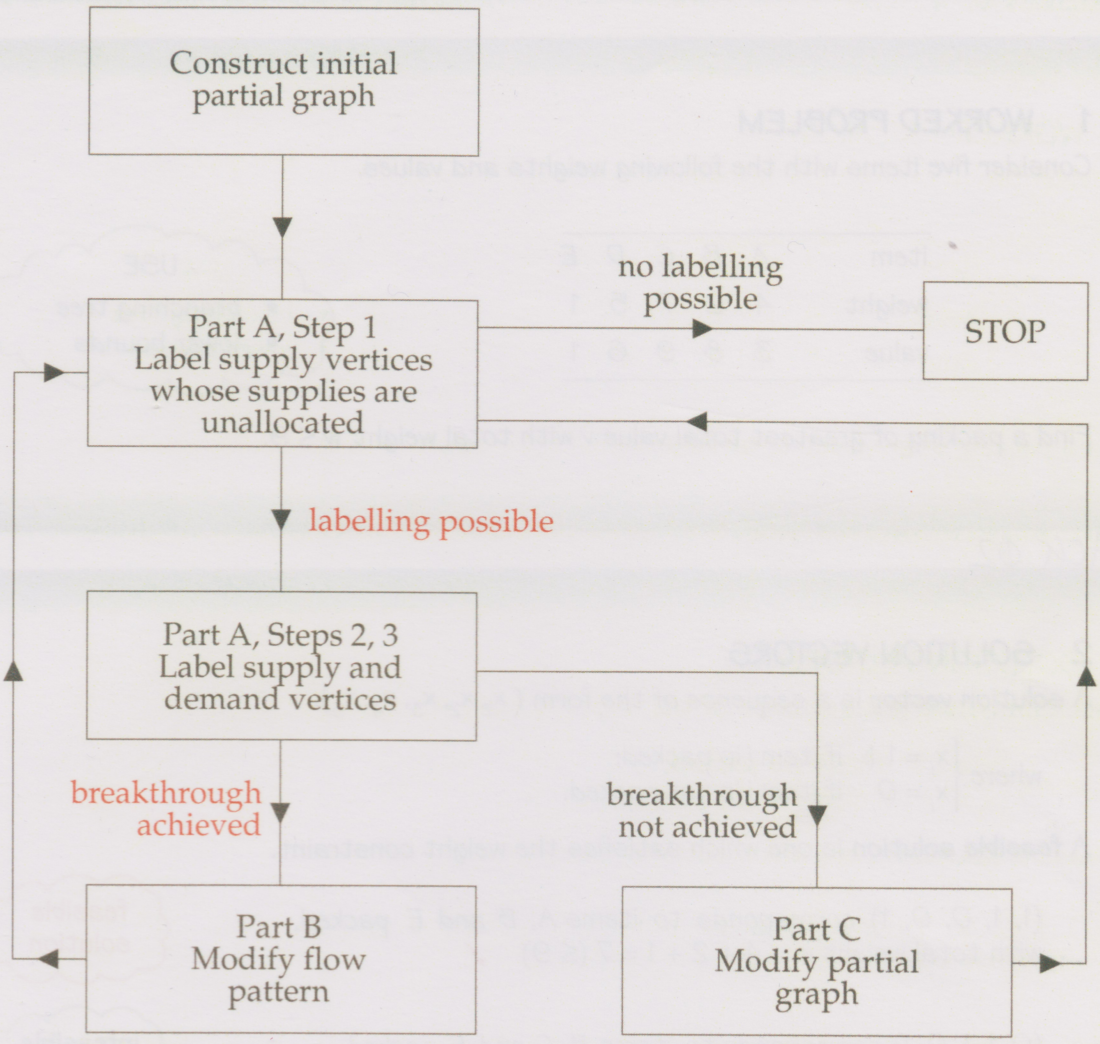
On the existing cost matrix:
draw a *horizontal* line through each labelled *supply* vertex;
draw a *vertical* line through each labelled *demand* vertex.

- find the smallest entry δ with only a *horizontal* line through it;
- *decrease* all entries with only a *horizontal* line through them by δ; *increase* the weight on the corresponding *supply* vertices by δ;
- *increase* all entries with only a *vertical* line through them by δ; *decrease* the weight on the corresponding *demand* vertices by δ.

Construct a revised partial graph.
(Remove any edge that now has a non-zero cost.)

Return to Part A.

4 FLOW CHART FOR THE ALGORITHM



problem in unit

TRACK Ø1

An algorithm for the knapsack problem

TRACK Ø1

1 WORKED PROBLEM

Consider five items with the following weights and values.

item	A	B	C	D	E
weight	4	2	7	5	1
value	3	8	9	6	1

USE
- branching tree
- lower bounds

Find a packing of greatest total value v with total weight $w \le 9$.

TRACK Ø2

2 SOLUTION VECTORS

A **solution vector** is a sequence of the form $(x_1, x_2, x_3, x_4, x_5)$,

$$\text{where} \begin{cases} x_i = 1 & \text{if item } i \text{ is packed;} \\ x_i = 0 & \text{if item } i \text{ is not packed.} \end{cases}$$

A **feasible solution** is one which satisfies the weight constraint.

(1, 1, 0, 0, 1) corresponds to items A, B and E packed, with total weight $w = 4 + 2 + 1 = 7$ (≤ 9) ✓

feasible solution

(0, 1, 1, 0, 1) corresponds to items B, C and E packed, with total weight $w = 2 + 7 + 1 = 10$ (> 9) ✗

infeasible solution

TRACK Ø3

3 BRANCHING IDEA

For example:

(0, 1, 0, 0, 0) → (0, 1, 1, 0, 0)
(0, 1, 0, 0, 0) → (0, 1, 0, 1, 0)
(0, 1, 0, 0, 0) → (0, 1, 0, 0, 1)

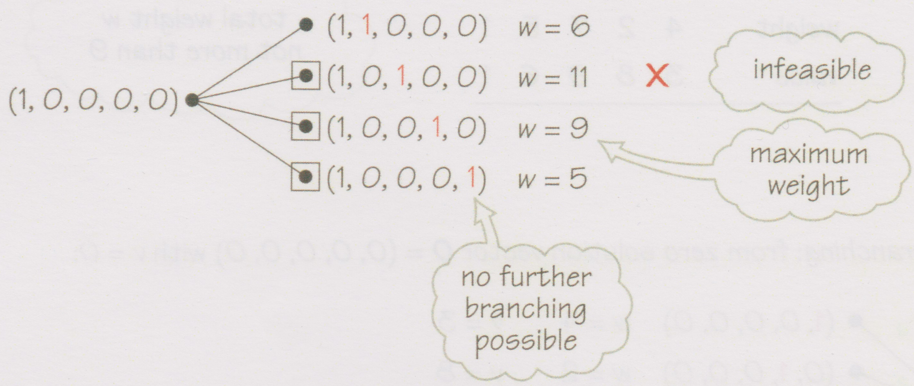
add a 1 to the right of this 1

these have one more item than the previous solution vector

TRACK Ø4

4 DECIDING HOW TO BRANCH

For example:



Next step: branch out from (1, 1, 0, 0, 0).

TRACK Ø5

5 OUTLINE OF ALGORITHM

zero vector is denoted by **0**

START with zero vector (0, 0, …, 0); feasible with value 0.
STORE (0, 0, …, 0) and value 0.

GENERAL STEP

- Branch from first solution from which branching is possible.
- Calculate total weight of each new solution.
- Calculate total value of each new *feasible* solution.
If there is a new feasible solution with value greater than the value stored, store the new solution vector and its value instead.
- Mark vertex with □ if it corresponds to:
  - □ a vector which equals or exceeds weight restriction;
  - □ a vector which ends in 1.

no further branching possible from ⊡

REPEAT the GENERAL STEP until no more branching is possible.

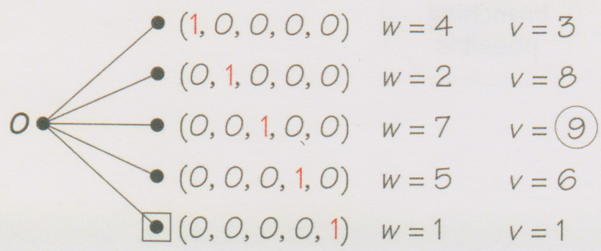
STOP Stored solution vector and value is optimum solution.

6 SOLUTION TO WORKED PROBLEM

item	A	B	C	D	E
weight	4	2	7	5	1
value	3	8	9	6	1

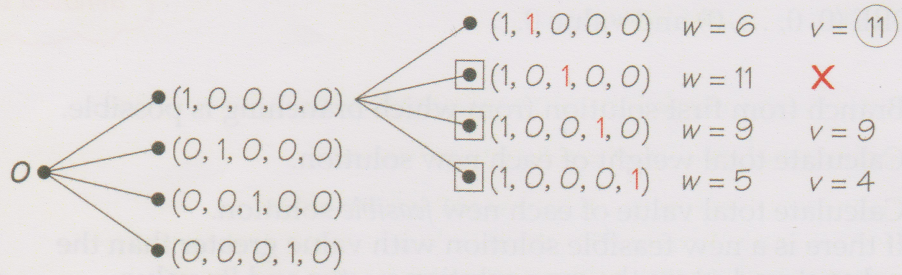
total weight w not more than 9

First branching: from zero solution vector $\mathbf{O} = (0, 0, 0, 0, 0)$ with $v = 0$:



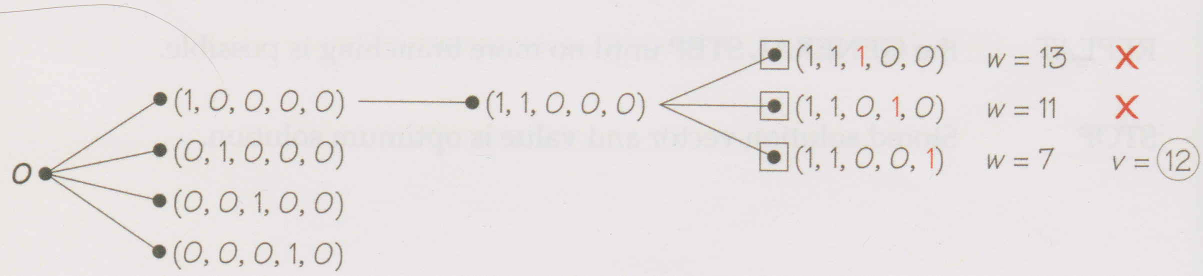
STORE $(0, 0, 1, 0, 0)$, $v = 9$

Second branching: from $(1, 0, 0, 0, 0)$:



STORE $(1, 1, 0, 0, 0)$, $v = 11$

Third branching: from $(1, 1, 0, 0, 0)$:

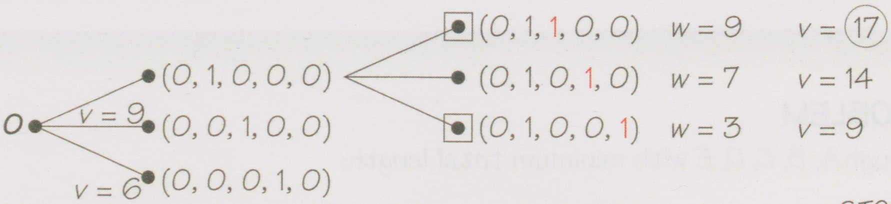


STORE $(1, 1, 0, 0, 1)$, $v = 12$

TRACK Ø6

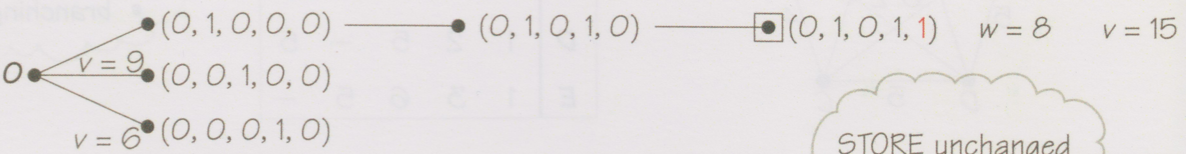
6 SOLUTION CONTINUED

Fourth branching: from (0, 1, 0, 0, 0):



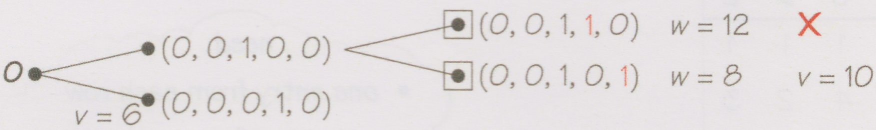
STORE (0, 1, 1, 0, 0), $v = 17$

Fifth branching: from (0, 1, 0, 1, 0):



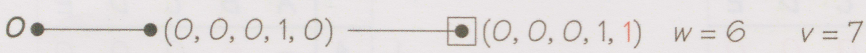
STORE unchanged

Sixth branching: from (0, 0, 1, 0, 0):



STORE unchanged

Seventh branching: from (0, 0, 0, 1, 0):



STORE unchanged

No further branching is possible: STOP.

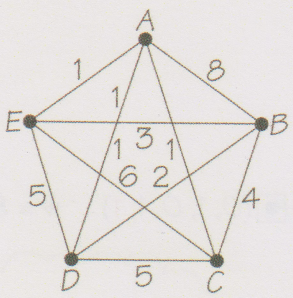
Solution vector is (0, 1, 1, 0, 0): items *B* and *C*, with value 17.

TRACK Ø7

An algorithm for the travelling salesman problem

1 WORKED PROBLEM

Find a 5-cycle through A, B, C, D, E with minimum total length:



	A	B	C	D	E
A	–	8	1	1	1
B	8	–	4	2	3
C	1	4	–	5	6
D	1	2	5	–	5
E	1	3	6	5	–

USE
- lower bounds
- branching tree

TRACK Ø8

2 GETTING LOWER BOUND FROM TABLE

	A	B	C	D	E
A	–	8	1	1	1
B	8	–	4	2	3
C	1	4	–	5	6
D	1	2	5	–	5
E	1	3	6	5	–

need
- one entry from each row
- one entry from each column
- no 2-, 3- or 4-cycles

		A	B	C	D	E
1	A	–	7	0	0	0
2	B	6	–	2	0	1
1	C	0	3	–	4	5
1	D	0	1	4	–	4
1	E	0	2	5	4	–

lower bound is 1 + 2 + 1 + 1 + 1 = 6

		0	1	0	0	0
		A	B	C	D	E
1	A	–	6	0	0	0
2	B	6	–	2	0	1
1	C	0	2	–	4	5
1	D	0	0	4	–	4
1	E	0	1	5	4	–

new lower bound is 6 + 1 = 7

TRACK Ø9

3 DECIDING HOW TO BRANCH

Consider edges with zero weight:

try to get maximum increase in lower bound

	A	B	C	D	E
A	–	6	0	0	0
B	6	–	2	0	1
C	0	2	–	4	5
D	0	0	4	–	4
E	0	1	5	4	–

exclude AC?

				2		
		A	B	C	D	E
0	A	–	6	X	0	0
	B	6	–	2	0	1
	C	0	2	–	4	5
	D	0	0	4	–	4
	E	0	1	5	4	–

lower bound increases by $0 + 2 = 2$

	A	B	C	D	E
A	–	6	0	0	0
B	6	–	2	0	1
C	0	2	–	4	5
D	0	0	4	–	4
E	0	1	5	4	–

exclude BD?

					0	
		A	B	C	D	E
	A	–	6	0	0	0
1	B	6	–	2	X	1
	C	0	2	–	4	5
	D	0	0	4	–	4
	E	0	1	5	4	–

lower bound increases by $1 + 0 = 1$

Label each zero with possible increase in lower bound.

Select an edge whose exclusion gives maximum increase in lower bound.

	A	B	C	D	E
A	–	6	0^2	0^0	0^1
B	6	–	2	0^1	1
C	0^2	2	–	4	5
D	0^0	0^1	4	–	4
E	0^1	1	5	4	–

maximum increase in lower bound arises from excluding AC

TRACK 10

4 CARRYING OUT BRANCHING

	A	B	C	D	E
A	–	6	0^2	0^0	0^1
B	6	–	2	0^1	1
C	0^2	2	–	4	5
D	0^0	0^1	4	–	4
E	0^1	1	5	4	–

select AC

lower bound 7

include AC
(so exclude CA)

delete row A and column C

cross out CA

	A	B	D	E
B	6	–	0	1
C	X	2	4	5
D	0	0	–	4
E	0	1	4	–

exclude AC

cross out AC

	A	B	C	D	E
A	–	6	X	0	0
B	6	–	2	0	1
C	0	2	–	4	5
D	0	0	4	–	4
E	0	1	5	4	–

TRACK 11

5 OUTLINE OF ALGORITHM

START with a given $n \times n$ table of distances, corresponding to a complete weighted graph with n vertices.

Carry out the initial row and column reduction, and calculate the initial lower bound.

GENERAL STEP

- Consider all the edges with zero weight, and choose an allowable edge e whose *exclusion* leads to the *greatest increase* in lower bound; if there are several such edges, choose the first.
- Consider the consequences of including e and excluding e. Use row and column reduction to determine these consequences, in terms of increases in the lower bound. Choose the option which gives the *smaller* lower bound; if the lower bounds are equal, *include* the edge e. STORE the current list of included edges, and the current lower bound.
- Continue from the current position *unless* the chosen option has a lower bound greater than a previously eliminated option, in which case backtrack to the earlier position.

CHECK avoid cycles with fewer than n edges

REPEAT the GENERAL STEP until a cycle with n edges has been created.

STOP Stored list of edges is optimum solution.

TRACK 12

6 SOLUTION TO WORKED PROBLEM: FIRST BRANCHING

Consider edges with zero weight:

1	A	B	C	D	E
A	–	6	0^2	0^0	0^1
B	6	–	2	0^1	1
C	0^2	2	–	4	5
D	0^0	0^1	4	–	4
E	0^1	1	5	4	–

select AC

include AC

	A	B	D	E
B	6	–	0	1
C	X	2	4	5
D	0	0	–	4
E	0	1	4	–

reduce row C by 2

reduce column E by 1

		0	0	0	1
	2	A	B	D	E
0	B	6	–	0	0
2	C	X	0	2	2
0	D	0	0	–	3
0	E	0	1	4	–

new lower bound is 7 + 2 + 1 = 10

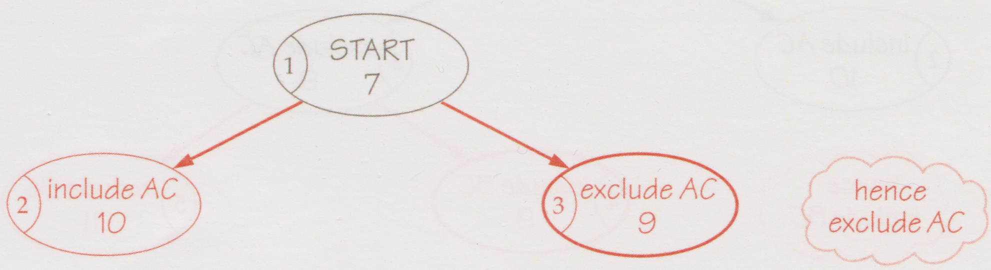
exclude AC

	A	B	C	D	E
A	–	6	X	0	0
B	6	–	2	0	1
C	0	2	–	4	5
D	0	0	4	–	4
E	0	1	5	4	–

reduce column C by 2

	0	0	2	0	0
3	A	B	C	D	E
A	–	6	X	0	0
B	6	–	0	0	1
C	0	2	–	4	5
D	0	0	2	–	4
E	0	1	3	4	–

new lower bound is 7 + 2 = 9



TRACK 13

7 SECOND BRANCHING

Consider edges with zero weight:

3	A	B	C	D	E
A	–	6	X	0^{0}	0^{1}
B	6	–	0^{2}	0^{0}	1
C	0^{2}	2	–	4	5
D	0^{0}	0^{1}	2	–	4
E	0^{1}	1	3	4	–

select BC
(lower bound increases by 2, the maximum possible)

include BC
(so exclude CB)

4	A	B	D	E
A	–	6	0	0
C	0	X	4	5
D	0	0	–	4
E	0	1	4	–

lower bound remains 9

exclude BC

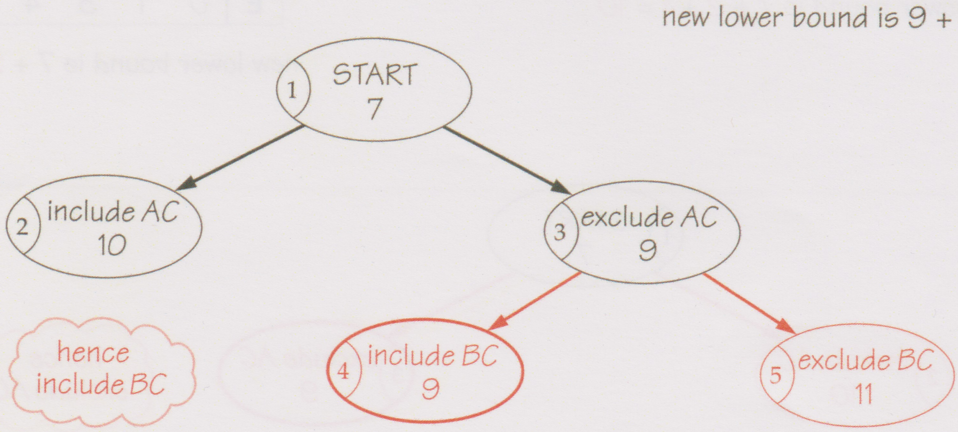
	A	B	C	D	E
A	–	6	X	0	0
B	6	–	X	0	1
C	0	2	–	4	5
D	0	0	2	–	4
E	0	1	3	4	–

reduce column C by 2

0 0 2 0 0

5	A	B	C	D	E
A	–	6	X	0	0
B	6	–	X	0	1
C	0	2	–	4	5
D	0	0	0	–	4
E	0	1	1	4	–

new lower bound is 9 + 2 = 11



TRACK 14

8 THIRD BRANCHING

Consider edges with zero weight:

4	A	B	D	E
A	–	6	0^4	0^4
C	0^4	X	4	5
D	0^0	0^1	–	4
E	0^1	1	4	–

select AD
(lower bound increases by 4, the maximum possible)

include AD
(so exclude DA)

	A	B	E
C	0	X	5
D	X	0	4
E	0	1	–

reduce column E by 4

	0	0	4
6	A	B	E
C	0	X	1
D	X	0	0
E	0	1	–

new lower bound is 9 + 4 = 13

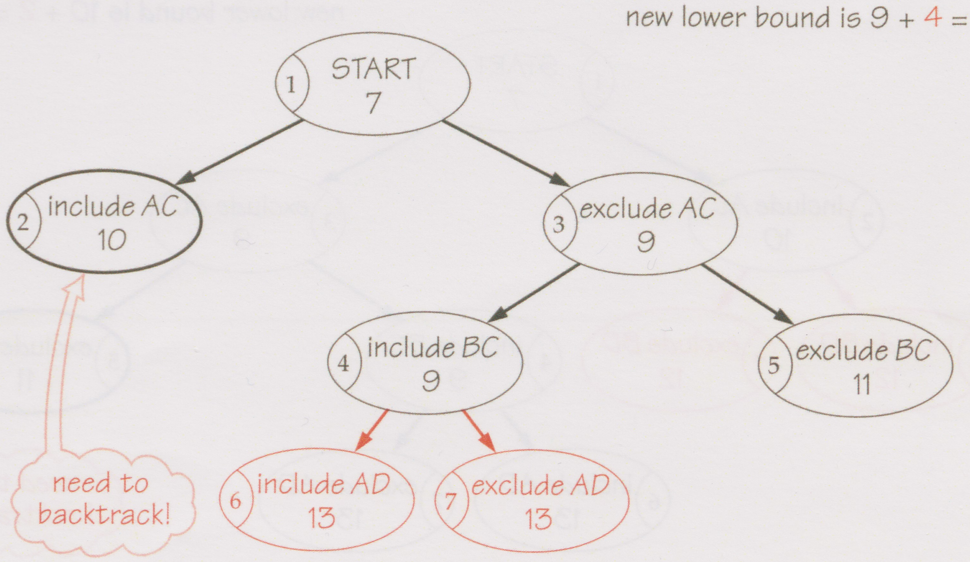
exclude AD

	A	B	D	E
A	–	6	X	0
C	0	X	4	5
D	0	0	–	4
E	0	1	4	–

reduce column D by 4

	0	0	4	0
7	A	B	D	E
A	–	6	X	0
C	0	X	0	5
D	0	0	–	4
E	0	1	0	–

new lower bound is 9 + 4 = 13



TRACK 15

9 FOURTH BRANCHING

Consider edges with zero weight:

2	A	B	D	E
B	6	–	0^{2}	0^{2}
C	X	0^{2}	2	2
D	0^{0}	0^{0}	–	3
E	0^{1}	1	4	–

select BD
(lower bound increases by 2, the maximum possible)

include BD
(so exclude DB)

	A	B	E
C	X	0	2
D	0	X	3
E	0	1	–

reduce column E by 2

8	A	B	E
	0	0	2
C	X	0	0
D	0	X	1
E	0	1	–

new lower bound is 10 + 2 = 12

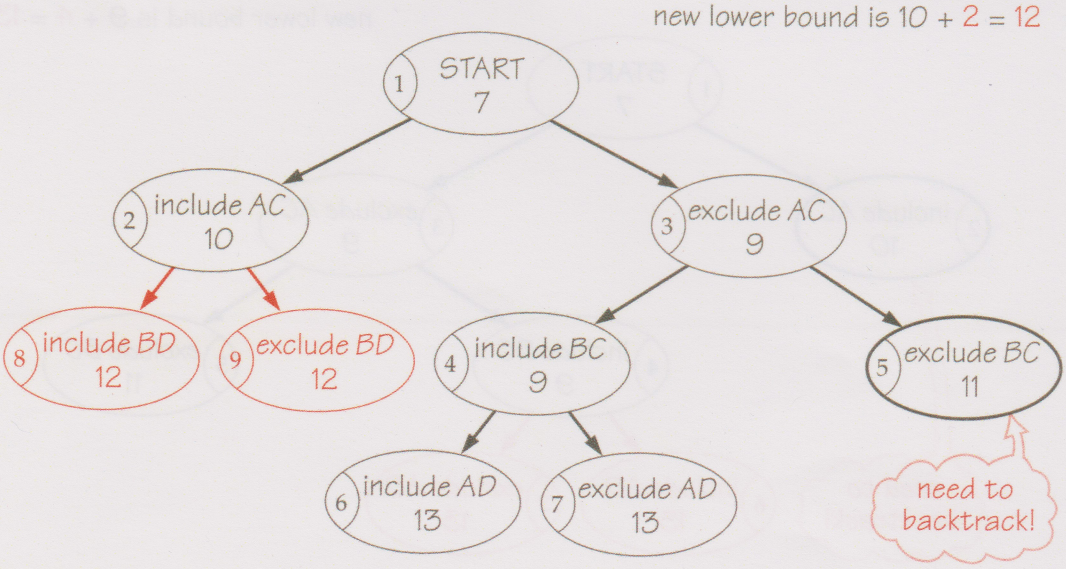
exclude BD

	A	B	D	E
B	6	–	X	0
C	X	0	2	2
D	0	0	–	3
E	0	1	4	–

reduce column D by 2

9	A	B	D	E
	0	0	2	0
B	6	–	X	0
C	X	0	0	2
D	0	0	–	3
E	0	1	2	–

new lower bound is 10 + 2 = 12



10 FIFTH BRANCHING

Consider edges with zero weight:

5	A	B	C	D	E
A	–	6	X	0^{0}	0^{1}
B	6	–	X	0^{1}	1
C	0^{2}	2	–	4	5
D	0^{0}	0^{1}	0^{1}	–	4
E	0^{1}	1	1	4	–

select CA
(lower bound increases by 2, the maximum possible)

include CA

	B	C	D	E
A	6	X	0	0
B	–	X	0	1
D	0	0	–	4
E	1	1	4	–

exclude CA

	A	B	C	D	E
A	–	6	X	0	0
B	6	–	X	0	1
C	X	2	–	4	5
D	0	0	0	–	4
E	0	1	1	4	–

reduce row E by 1

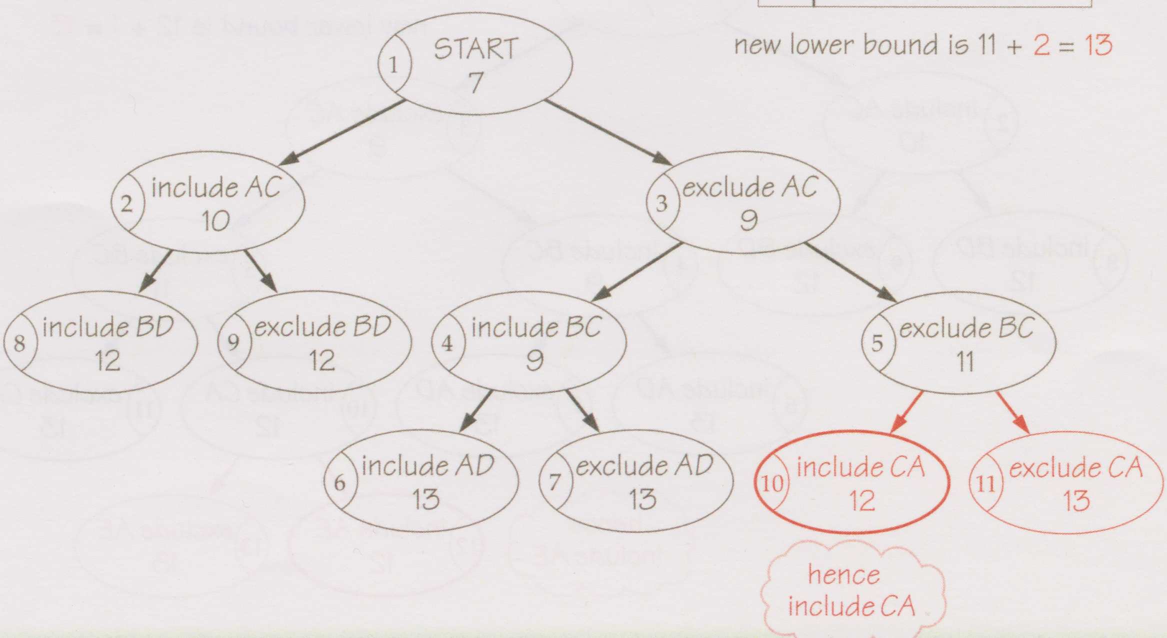
	10	B	C	D	E
0	A	6	X	0	0
0	B	–	X	0	1
0	D	0	0	–	4
1	E	0	0	3	–

new lower bound is 11 + 1 = 12

reduce row C by 2

	11	A	B	C	D	E
0	A	–	6	X	0	0
0	B	6	–	X	0	1
2	C	X	0	–	2	3
0	D	0	0	0	–	4
0	E	0	1	1	4	–

new lower bound is 11 + 2 = 13

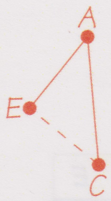


11 SIXTH BRANCHING

Consider edges with zero weight:

10	B	C	D	E
A	6	X	0^0	0^1
B	–	X	0^1	1
D	0^0	0^0	–	4
E	0^0	0^0	3	–

select AE
(lower bound increases by 1, the maximum possible)



include AE
(so exclude EC to avoid 3-cycle AECA)

12	B	C	D
B	–	X	0
D	0	0	–
E	0	X	3

lower bound remains 12

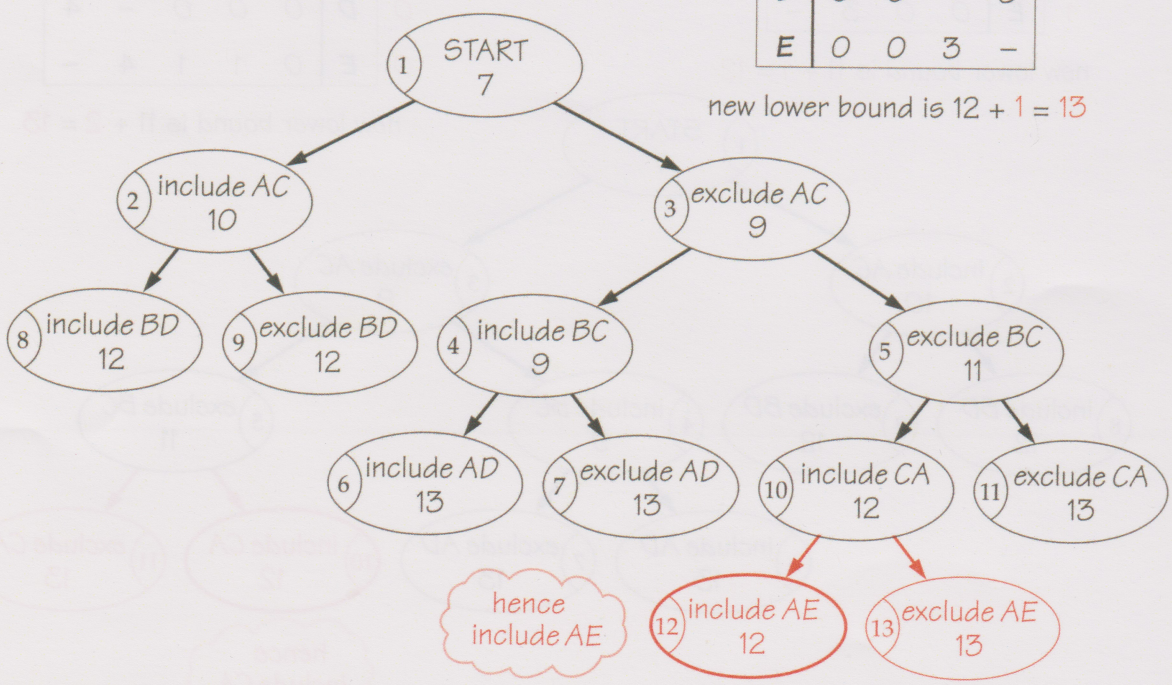
exclude AE

	B	C	D	E
A	6	X	0	X
B	–	X	0	1
D	0	0	–	4
E	0	0	3	–

reduce column E by 1

	0	0	0	1
13	**B**	**C**	**D**	**E**
A	6	X	0	X
B	–	X	0	0
D	0	0	–	3
E	0	0	3	–

new lower bound is 12 + 1 = 13



TRACK 18

12 FINAL BRANCHINGS

Consider edges with zero weight:

12	B	C	D
B	–	X	0^3
D	0^0	0^0	–
E	0^3	X	3

select BD
(lower bound increases by 3, the maximum possible)

include BD
(so exclude DB)

14	B	C
D	X	0
E	0	X

lower bound remains 12

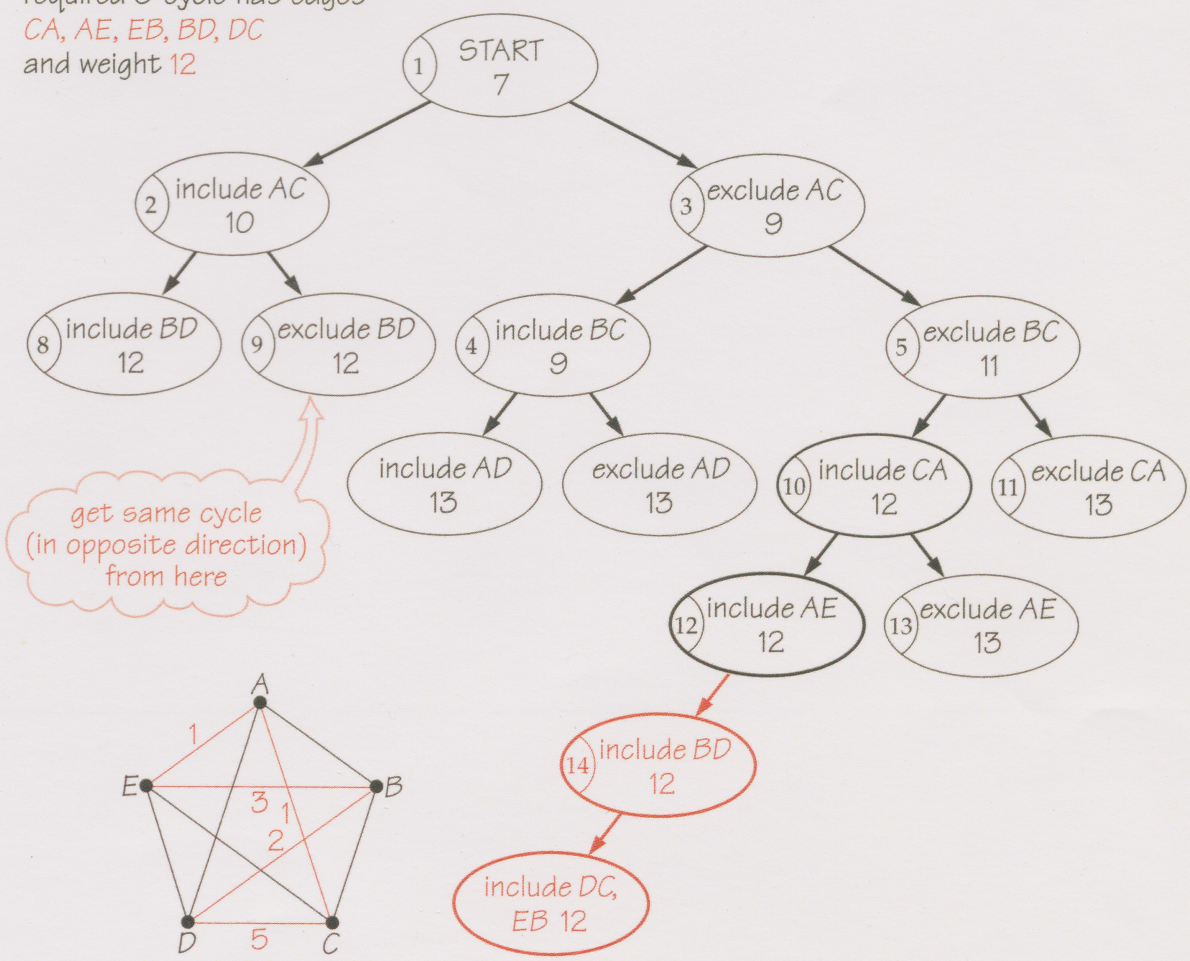
Include DC and EB
lower bound remains 12

exclude BD

15	B	C	D
B	–	X	X
D	0	0	–
E	0	X	3

not possible!
(no route out of B)

required 5-cycle has edges
CA, AE, EB, BD, DC
and weight 12



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